System Modeling

Petri Nets

Definition: A Petri net is a six-tuple $N = (P, T, A, w, x_0)$, where

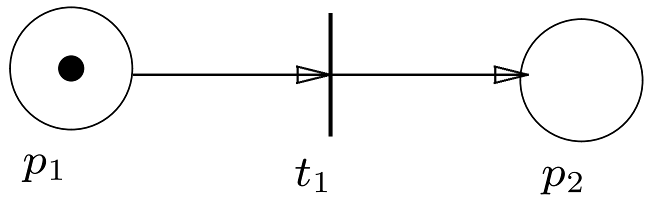
P	is a finite set of places
T	is a finite set of transitions
A	is a set of arcs, $A \subseteq (P \times T) \cup (T \times P)$
w	is a weight function, $w : A \rightarrow \mathbb{N}$
$\vec{x}_0$	is an initial marking vector, $\vec{x}_0 \in \mathbb{N}^{\|P\|}$

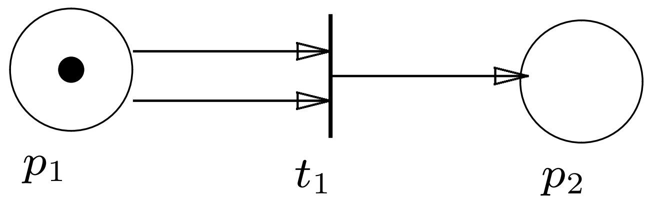
Definition: Let $N = (P, T, A, w, \vec{x}_0)$ be a petri net. The set $I(t) = \{p \in P | (p, t) \in A\}$ is the set of input places of transition t. The set $O(t) = \{p \in P | (t, p) \in A\}$ is the set of output places of transition t.

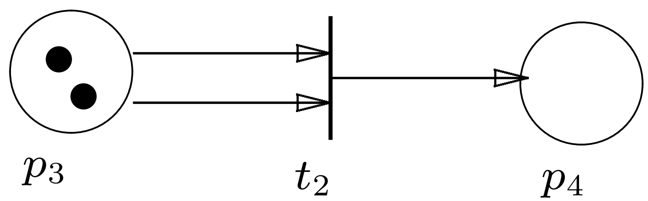
A transition t is enabled in state $\vec{x}$ if

$$x(p) \geq w(p, t) \; \forall p \in I(t).$$

Petri Net Examples







Petri Net Transition

Definition: Let $N = (P, T, A, w, \vec{x}_0)$ be a petri net with $P = \{p_0, \ldots, p_{n-1}\}$ and $\vec{x} = [x(p_0), \ldots, x(p_{n-1})]$ be a marking for the n places. The the transition function $G : (\mathbb{N}^n \times T) \rightarrow \mathbb{N}^n$ is defined as follows

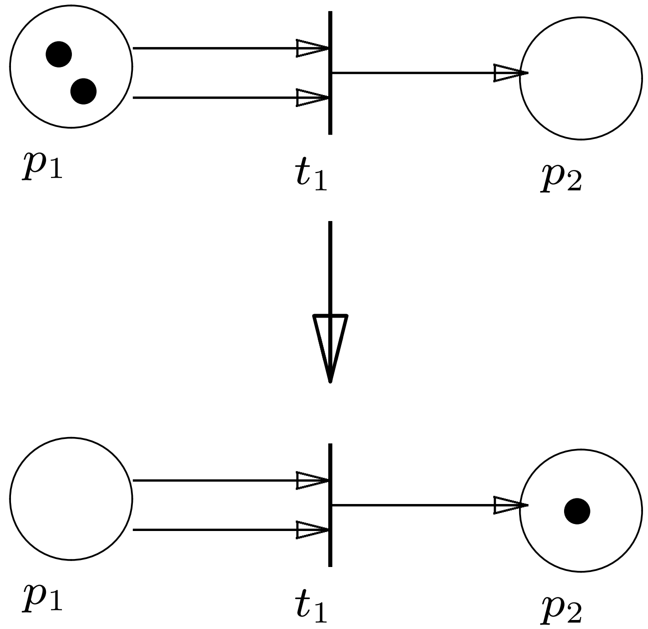
$$G(\vec{x}, t) = \begin{cases} \vec{x}' & \text{if } x(p) \geq w(p,t) \; \forall p \in I(t) \\ \vec{x} & \text{otherwise} \end{cases}$$

with

$$\vec{x}' = [x'(p_0), \ldots, x'(p_{n-1})]$$

$$x'(p_i) = x(p_i) - w(p_i, t) + w(t, p_i) \text{ for } 0 \leq i < n$$

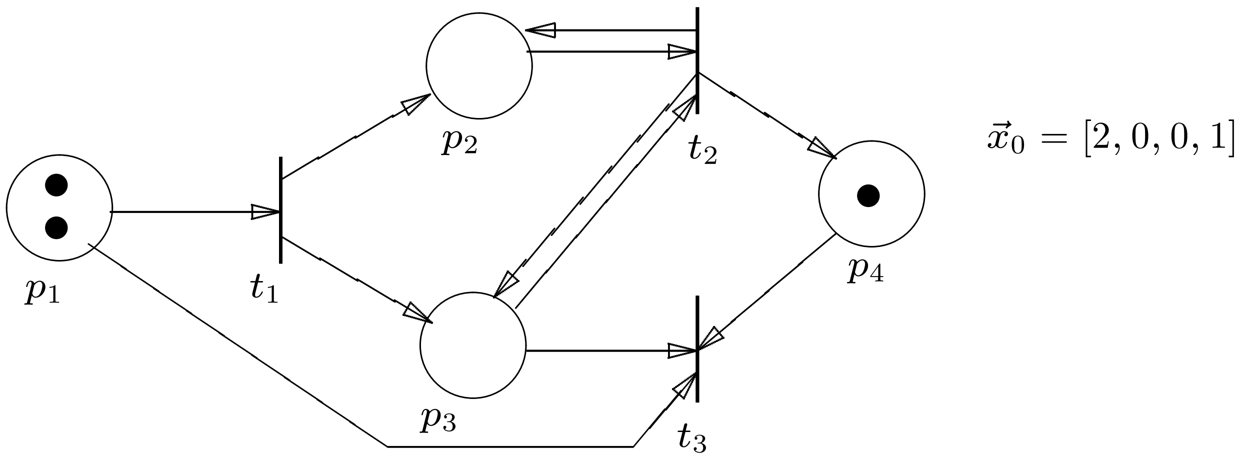
Firing of a Transition



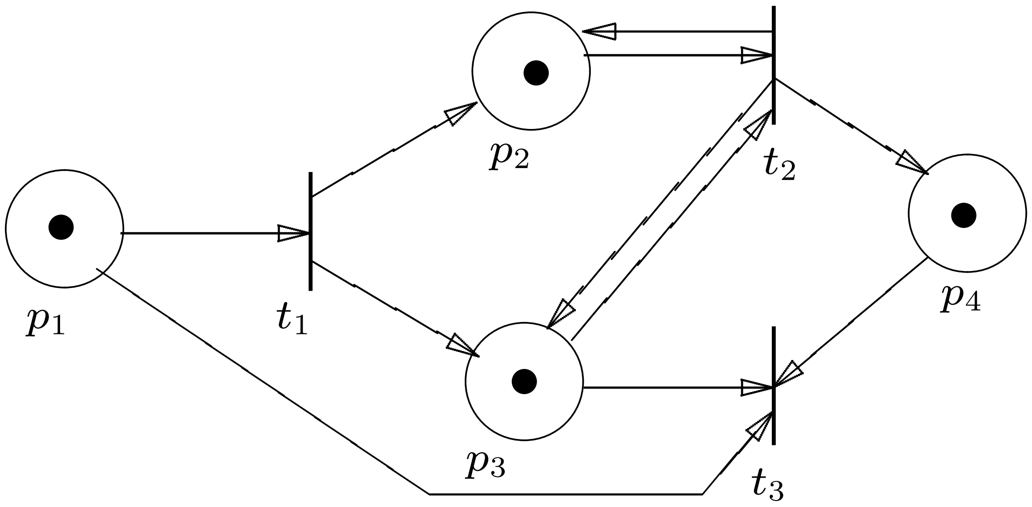
Petri Net Dynamics Example - 1

Petri net $N = (P, T, A, w, \vec{x}_0)$ with

$$
\begin{aligned}
P &= \{p_1, p_2, p_3, p_4\} \\
T &= \{t_1, t_2, t_3\} \\
A &= \{(p_1, t_1), (p_1, t_3), (p_2, t_2), (p_3, t_2), (p_3, t_3), (p_4, t_3), \\
&\quad (t_1, p_2), (t_1, p_3), (t_2, p_2), (t_2, p_3), (t_2, p_4)\} \\
w(a) &= 1 \ \forall a \in A \\
\vec{x}_0 &= [2, 0, 0, 1],
\end{aligned}
$$

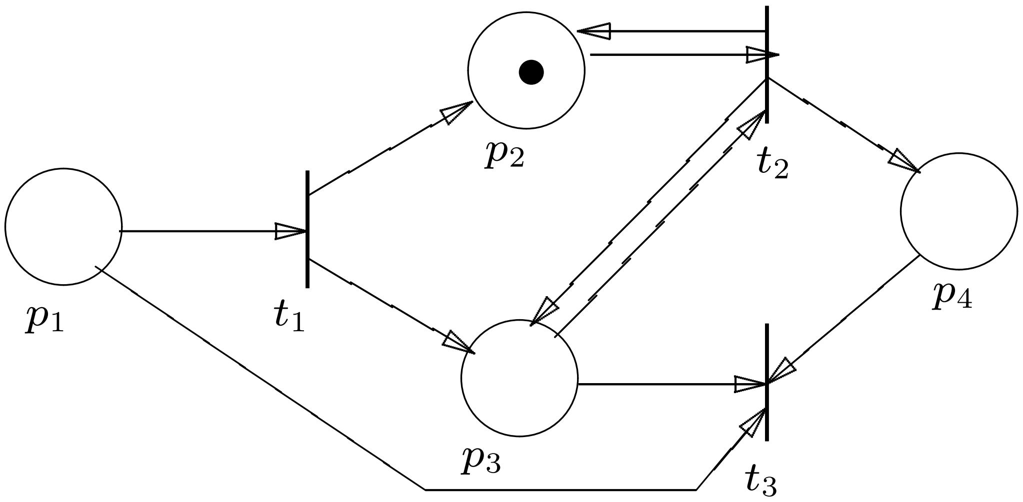


Petri Net Dynamics Example - 2



$$\vec{x}_0 = [2, 0, 0, 1]$$
$$\downarrow t_1$$
$$\vec{x}_1 = [1, 1, 1, 1]$$

Petri Net Dynamics Example - 3



$$\vec{x}_0 = [2, 0, 0, 1]$$
$$\downarrow t_1$$
$$\vec{x}_1 = [1, 1, 1, 1]$$
$$\downarrow t_3$$
$$\vec{x}_2 = [0, 1, 0, 0]$$

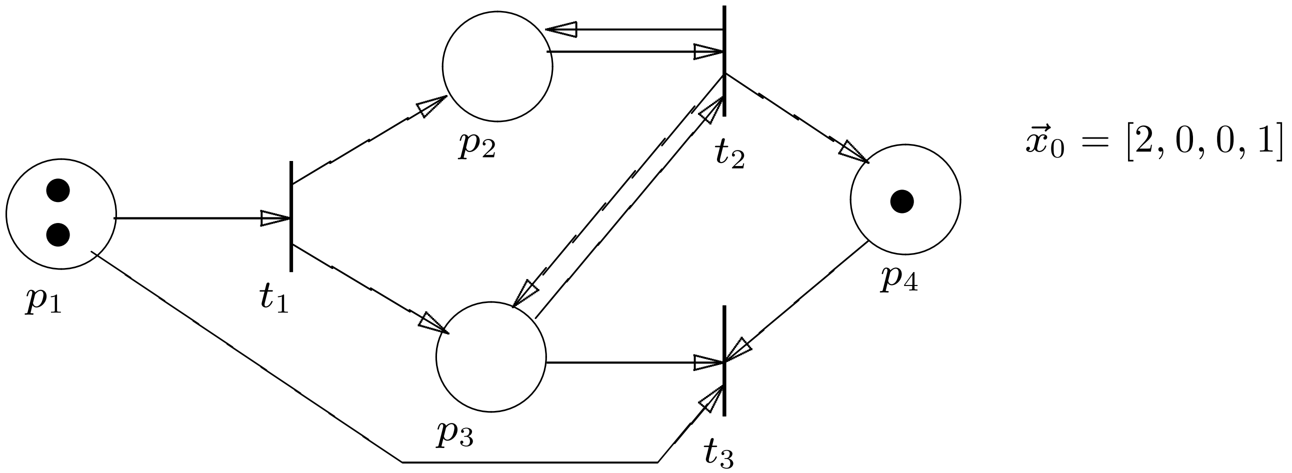
The Reachability Set

Definition: For a Petri net $N = (P, T, A, w, \vec{x}_0)$ and a given state $\vec{x}$, a state $\vec{y}$ is immediately reachable from $\vec{x}$ if there exists a transition $t \in T$ such that $G(\vec{x}, t) = \vec{y}$.

The reachability set $R(\vec{x})$ is the smallest set of states defined by

1. $\vec{x} \in R(\vec{x})$
2. If $\vec{y} \in R(\vec{x})$ and $z = G(\vec{y}, t)$ for some $t \in T$, then $\vec{z} \in R(\vec{x})$.

Reachability Set Example



$$
\begin{aligned}
R(\vec{x}_0) &= R_1 \cup R_2 \cup R_3 \cup R_4 \\
R_1 &= \{\vec{x}_0\} \\
R_2 &= \{\vec{y} \mid \vec{y} = [1, 1, 1, n], n \geq 1\} \\
R_3 &= \{\vec{y} \mid \vec{y} = [0, 2, 2, n], n \geq 1\} \\
R_4 &= \{\vec{y} \mid \vec{y} = [0, 1, 0, n], n \geq 0\}
\end{aligned}
$$

Firing Vector and Incidence Matrix

Definition: Let $N = (P, T, A, w, \vec{x}_0)$ be a petri net with $P = \{p_1, \ldots, p_n\}$ and $T = \{t_1, \ldots, t_m\}$. A firing vector $\vec{u} = [0, \ldots, 0, 1, 0, \ldots, 0]$ is a vector of length m where entry $j, 1 \geq j \geq m$, corresponds to transition t_j. All entries of the vector are 0 but one, where it has a value of 1. If entry j is 1, transition t_j fires.

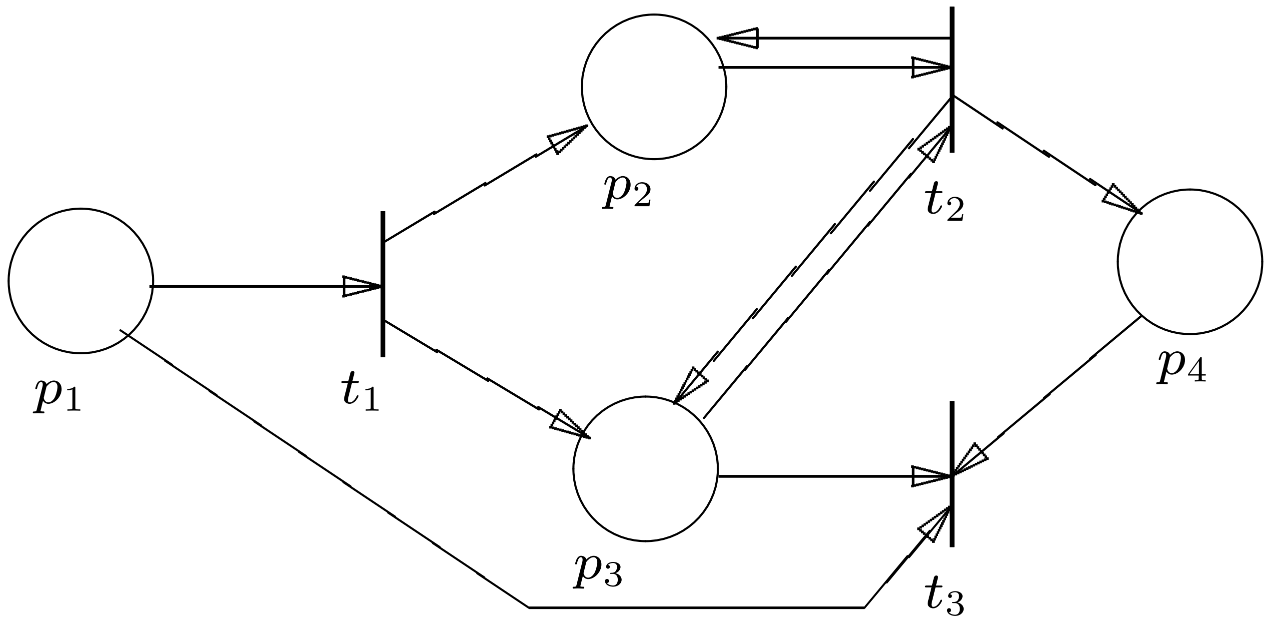
The incidence matrix $\boldsymbol{\mathcal{A}}$ is an $m \times n$ matrix whose (j, i) entry is

$$a_{j,i} = w(t_j, p_i) - w(p_i, t_j)$$

A state equation can be written as

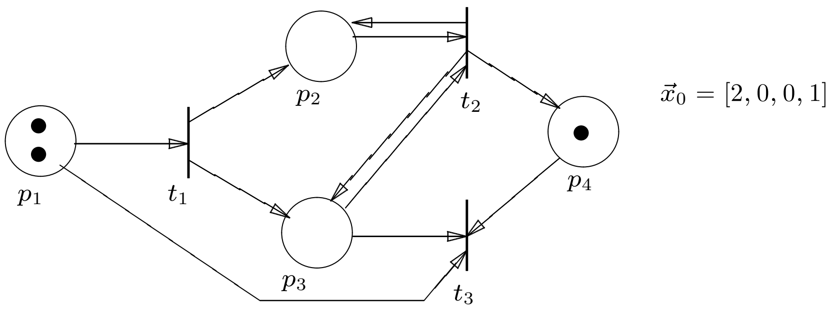
$$\vec{x}' = \vec{x} + \vec{u}\boldsymbol{\mathcal{A}}$$

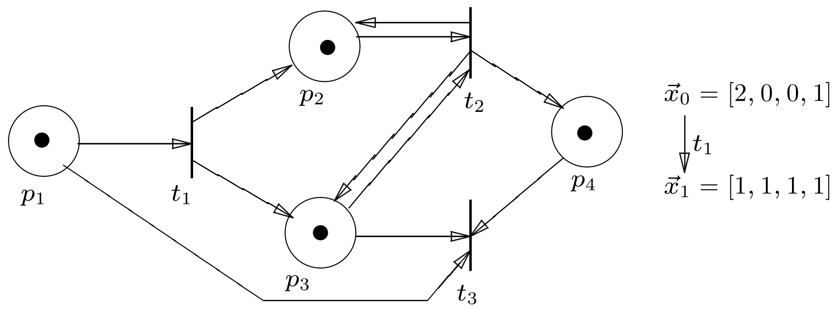
Incidence Matrix Example



$$\boldsymbol{\mathcal{A}} = \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix},$$

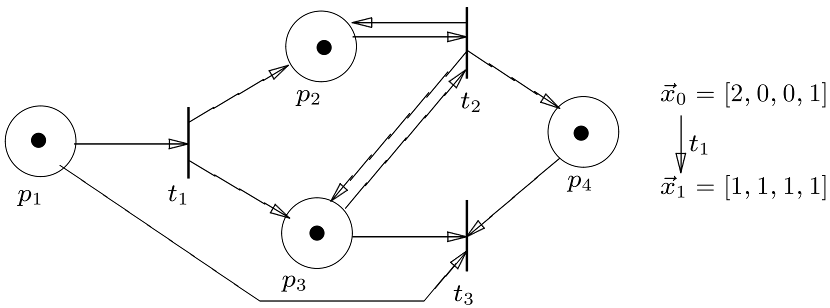
The Evaluation of State Equations

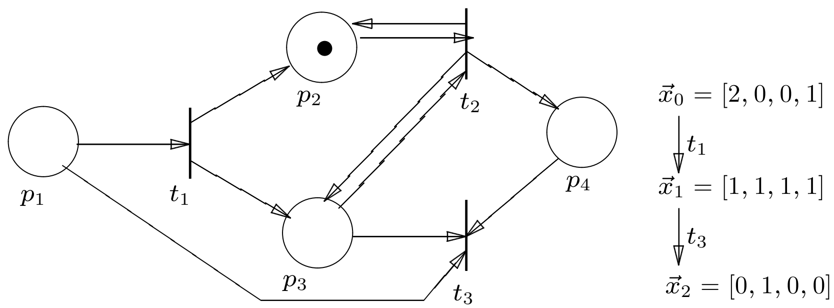




$$
\begin{aligned}
\vec{x}_1 &= \vec{x}_0 + \vec{u}_1 \boldsymbol{\mathcal{A}} \\
&= [2,0,0,1] + [1,0,0] \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix} \\
&= [2,0,0,1] + [-1+0+0, 1+0+0, 1+0+0, 0+0+0] \\
&= [2,0,0,1] + [-1,1,1,0] = [1,1,1,1]
\end{aligned}
$$

The Evaluation of State Equations - cont'd





$$\begin{aligned}
\vec{x}_2 &= \vec{x}_1 + \vec{u}_2 \boldsymbol{\mathcal{A}} \\
&= [1,1,1,1] + [0,0,1] \begin{bmatrix} -1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -1 & -1 \end{bmatrix} \\
&= [1,1,1,1] + [-1,0,-1,-1] = [0,1,0,0]
\end{aligned}$$

The Evaluation of a Transition Sequence

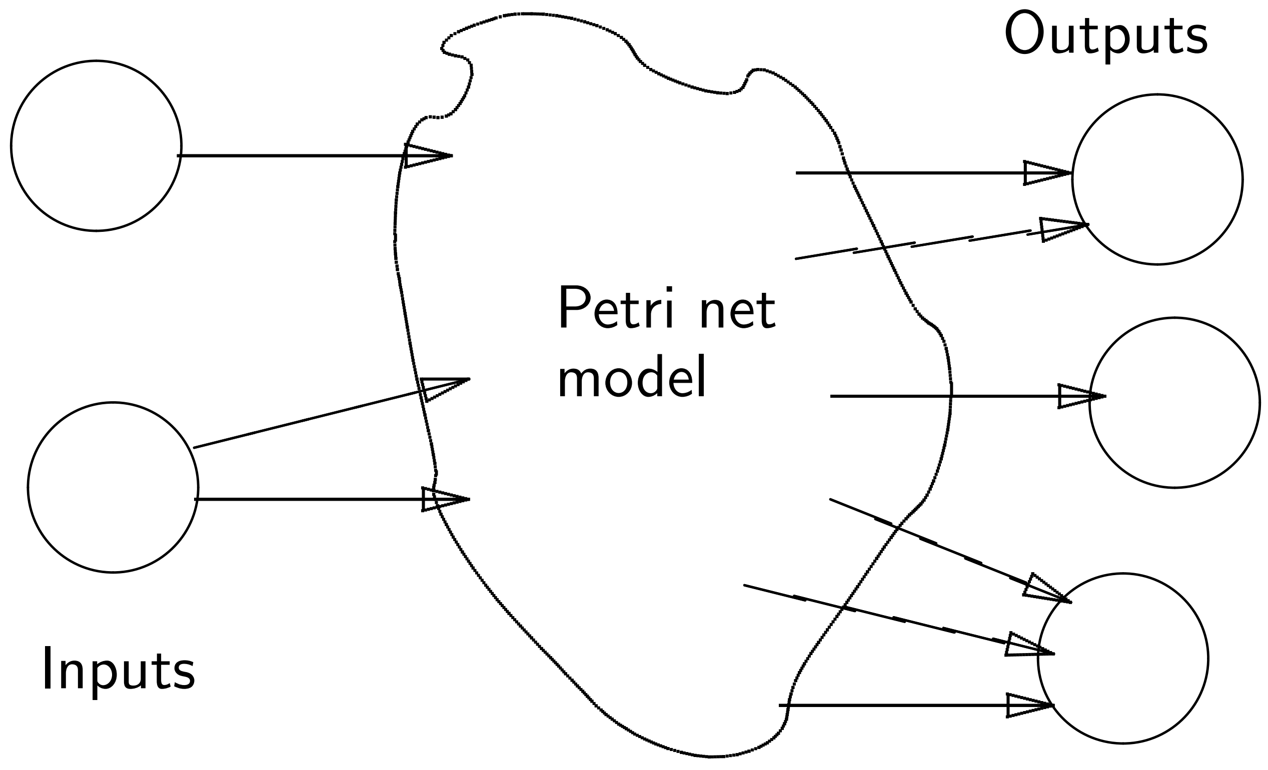
$N = (P, T, A, w, \vec{x}_0)$ is a Petri net;
$T' = \langle t_1, t_2, \ldots, t_i, \ldots, t_n \rangle, t_i \in T$ a sequence of n transitions with $\vec{u}_t$ the transition vector for t.
The state after firing of all transitions in T' is

$$\vec{x}_n = \vec{x}_0 + (\sum_{t \in T'} \vec{u}_t)\boldsymbol{\mathcal{A}}$$

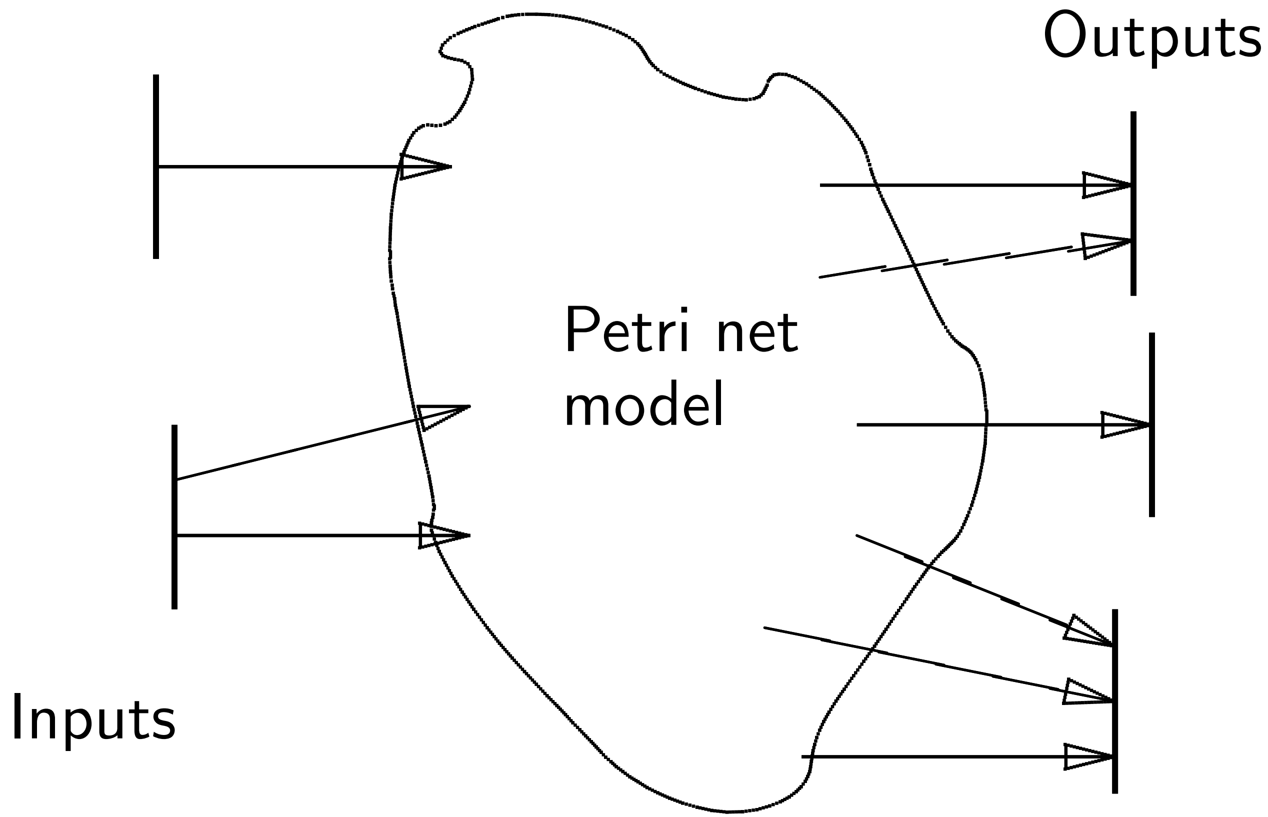
provided that for all $t_i \in T'$, t_i is enabled in state

$$\vec{x}_{i-1} = \vec{x}_0 + (\sum_{t \in \langle t_1, \ldots, t_{i-1} \rangle} \vec{u}_t)\boldsymbol{\mathcal{A}}.$$

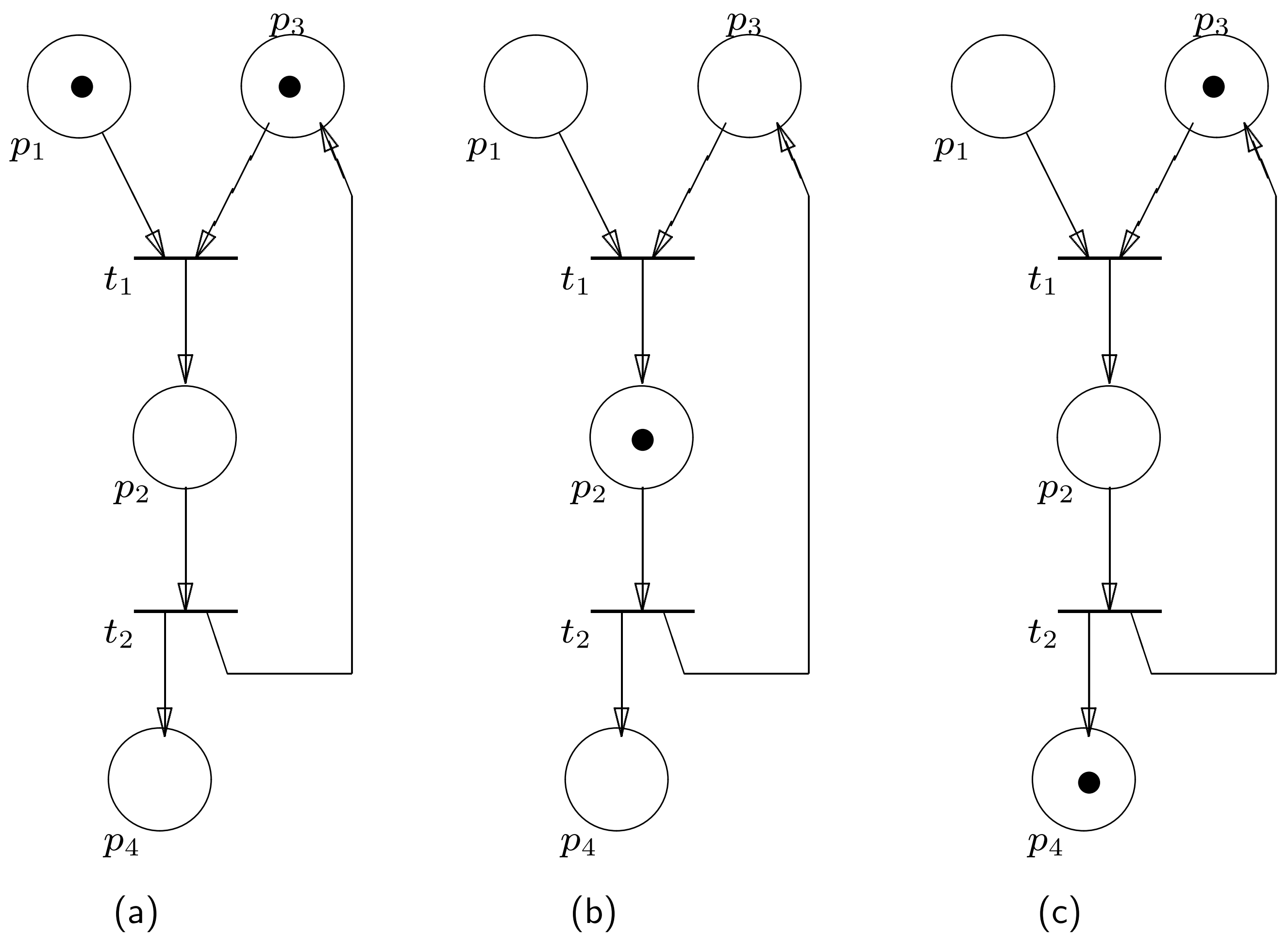
I/O Modeled as Places



I/O Modeled as Transitions

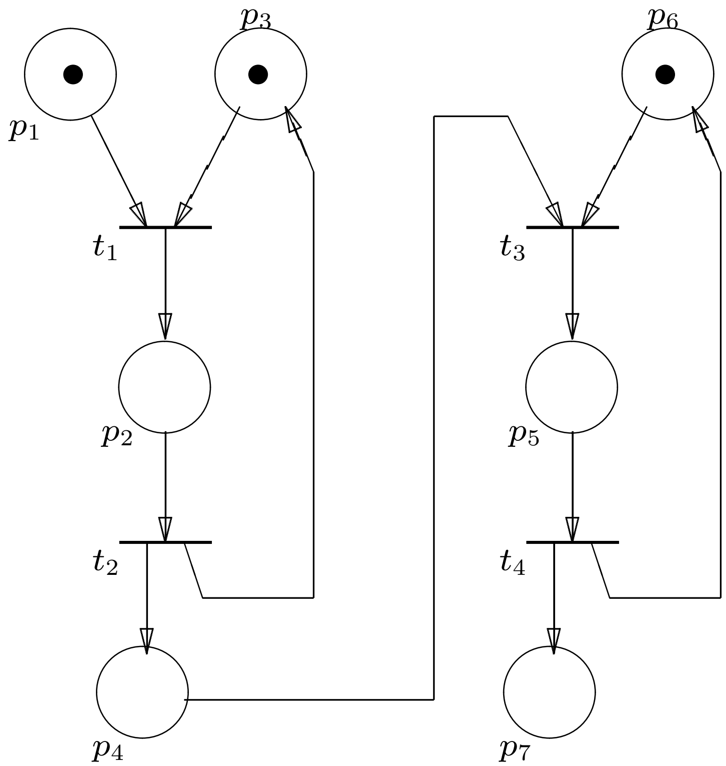


A Server Modeled as Petri Net



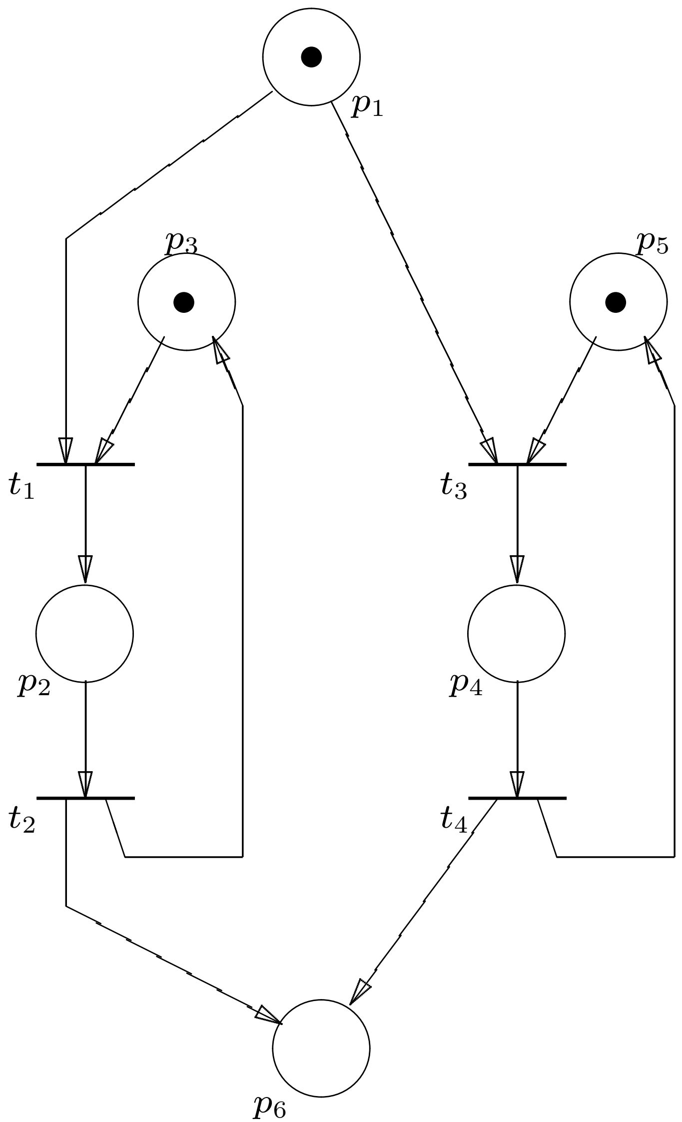
Customers arrive at input p_1 and depart at output p_4.

Sequential Composition of two Servers



Customers arrive at input p_1 and depart at output p_7.

Parallel Composition of two Servers



Customers arrive at input p_1 and depart at output p_6.

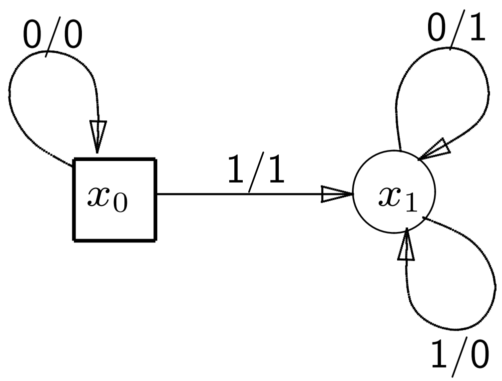
A Finite State Machine Modeled as Petri Net

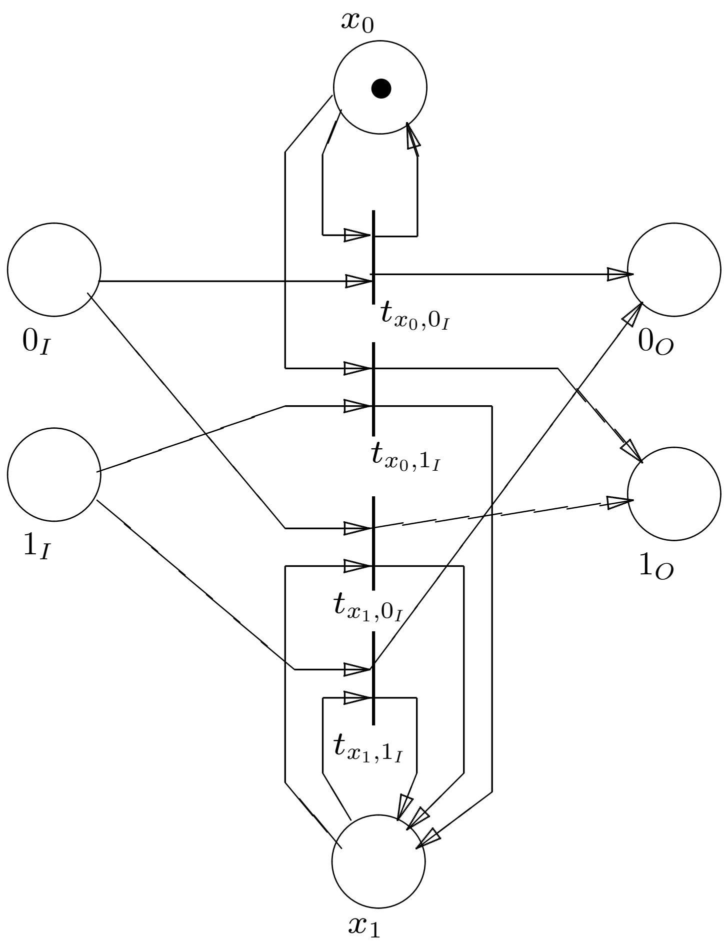
FSM $M = (\Sigma, \Delta, X, x_0, g, f)$ with mutually exclusive sets Σ and Δ.
An equivalent Petri net is $N = (P, T, A, \vec{y}_0)$ with

$$\begin{aligned} P &= X \cup \Sigma \cup \Delta \\ T &= \{t_{x,a} | x \in X, a \in \Sigma\} \\ A &= I(t_{x,a}) \cup O(t_{x,a}) \;\; \forall t_{x,a} \in T \\ I(t_{x,a}) &= \{x, a\} \\ O(t_{x,a}) &= \{g(x,a), f(x,a)\} \\ \vec{y}_0 &= [1, 0, \ldots, 0] \end{aligned}$$

- Σ are input places;
- Δ are output places;
- X are internal places;
- Each (state,input) pair in M becomes a transition in N;
- Initial marking represents state x_0 and no input;

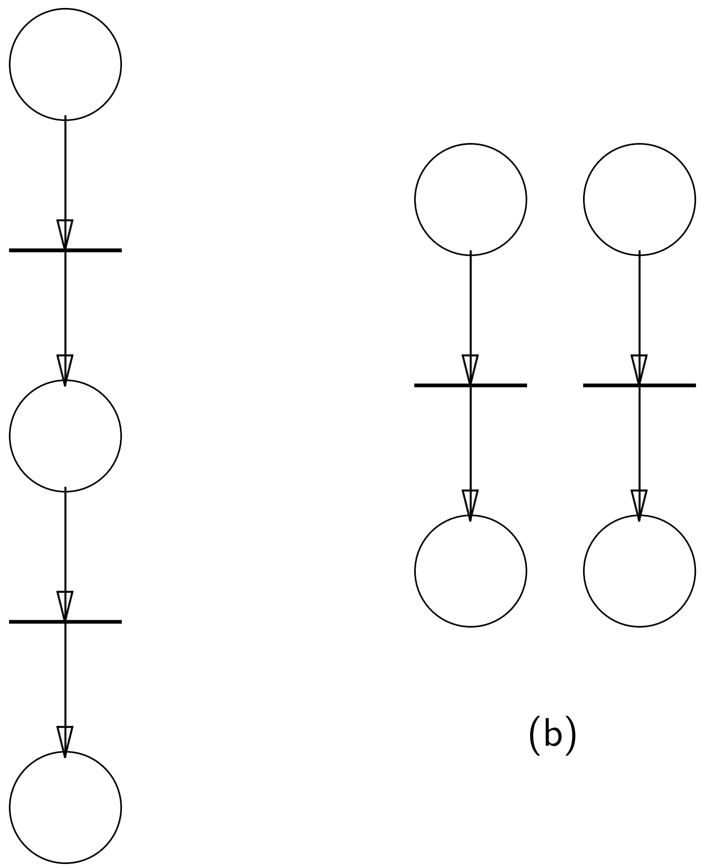
A FSM Modeled as Petri Net - Example





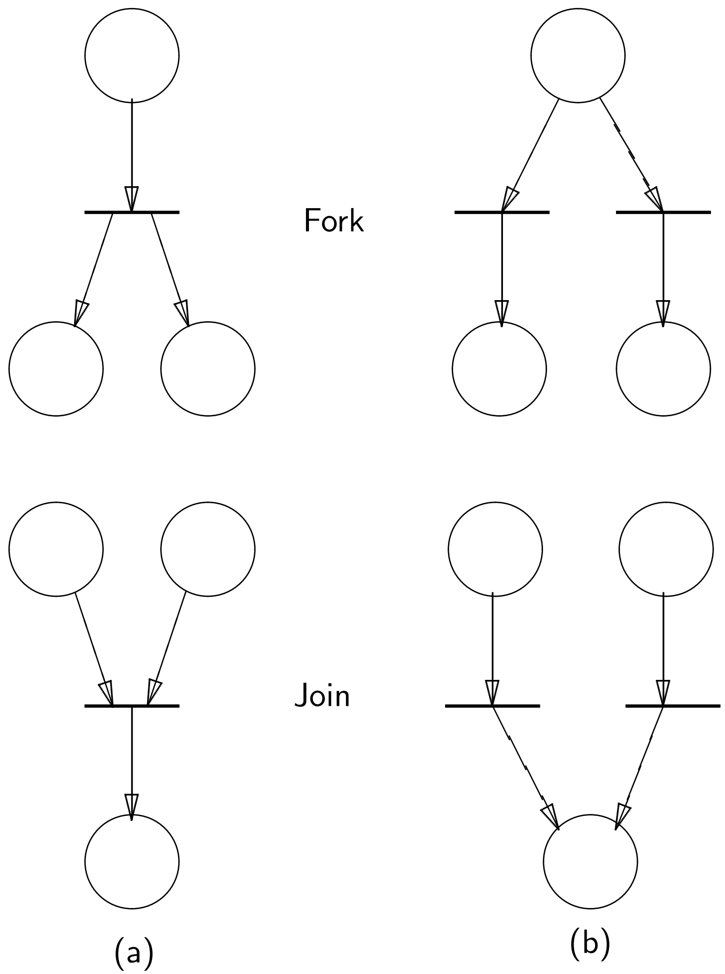
Computation of the two's complement of a binary number represented with the least significant bit first.

Sequence and Concurrency

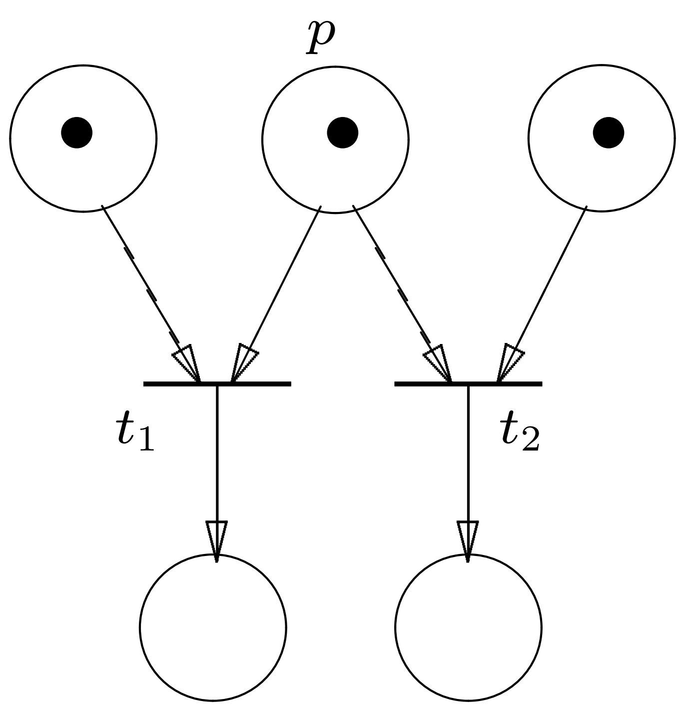


(a) (b)

Fork and Join



Conflict



The Mutual Exclusion Problem

```
read(x);
set x <- x + 1;
write(x);
```

Process A

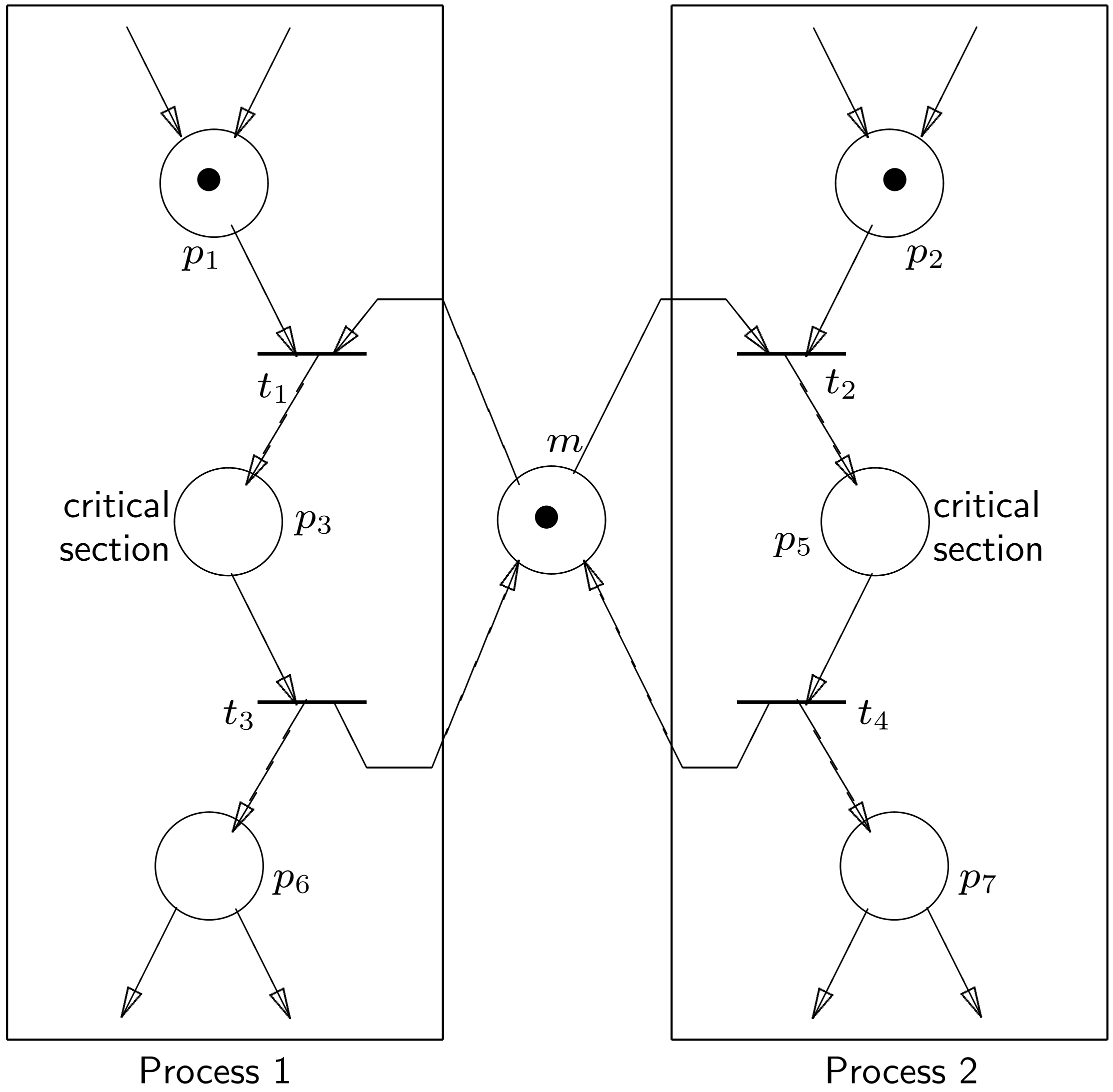
```
read(x);
set x <- x + 1;
write(x);
```

Process B

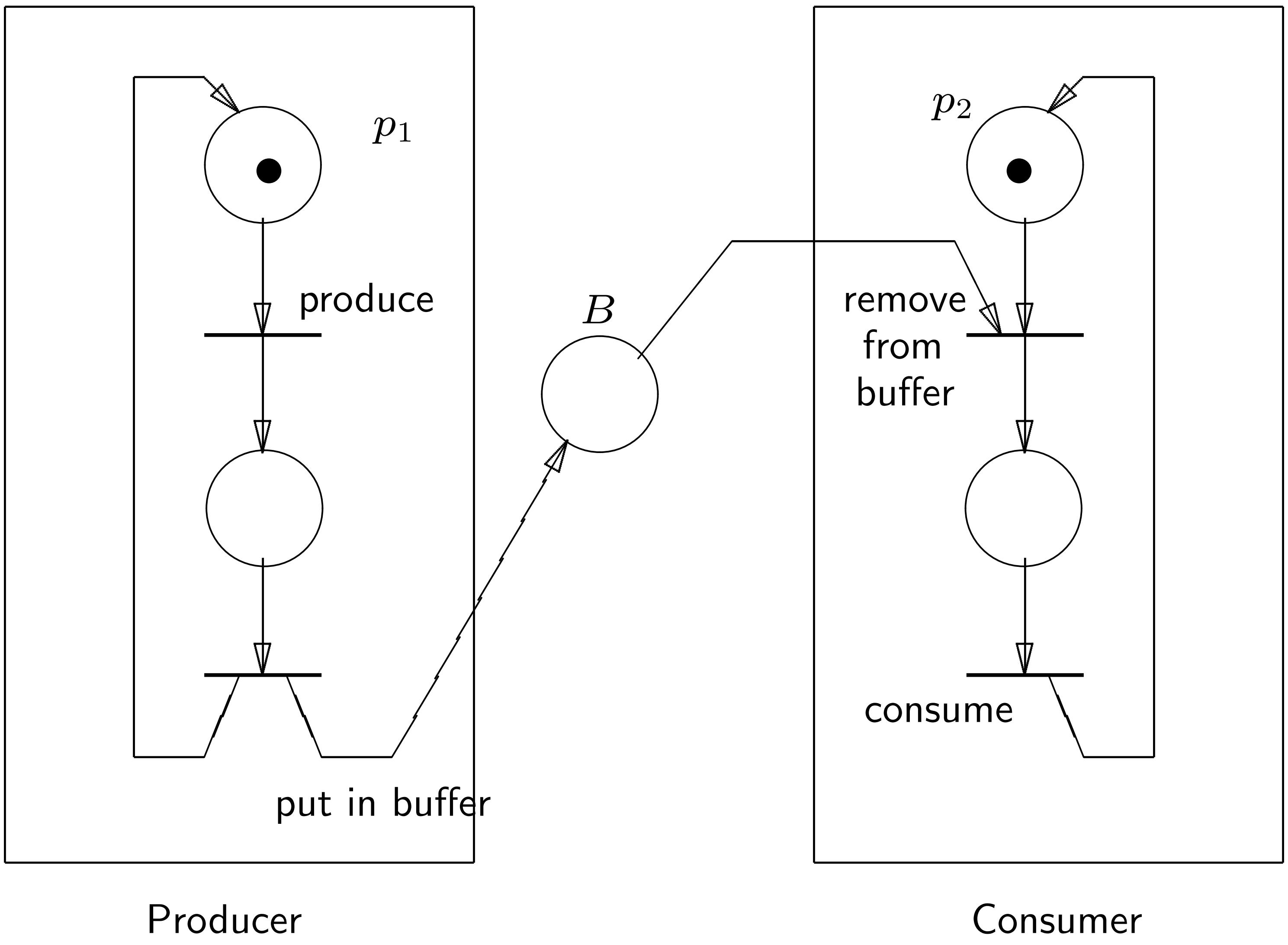
```
x <- 0;
A.read(x);
A.set x <- x + 1;
A.write(x);
B.read(x);
B.set x <- x + 1;
B.write(x);
x == 2
```

```
x <- 0;
A.read(x);
B.read(x);
A.set x <- x + 1;
A.write(x);
B.set x <- x + 1;
B.write(x);
x == 1
```

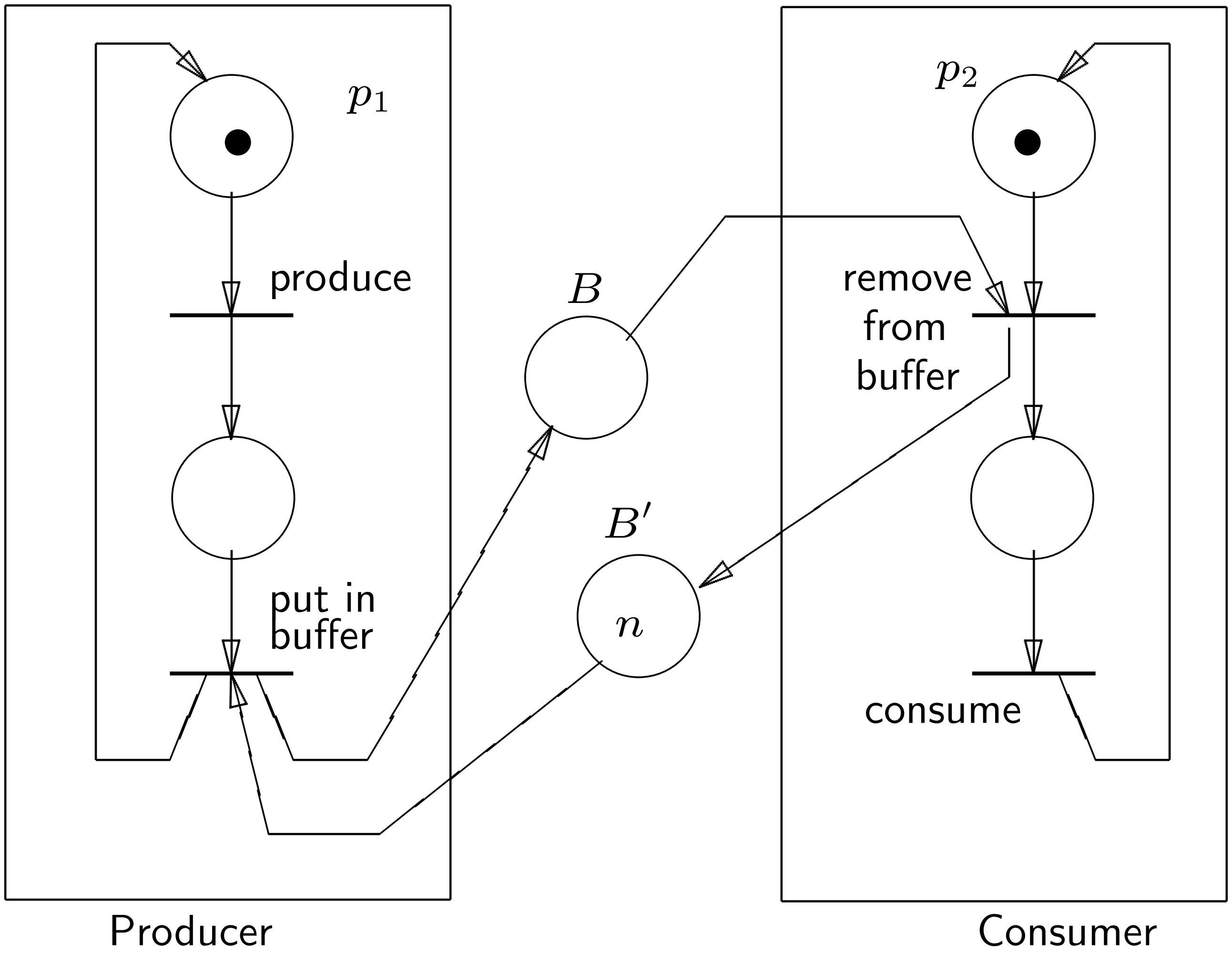
Mutual Exclusion Modeled with a Petri Net



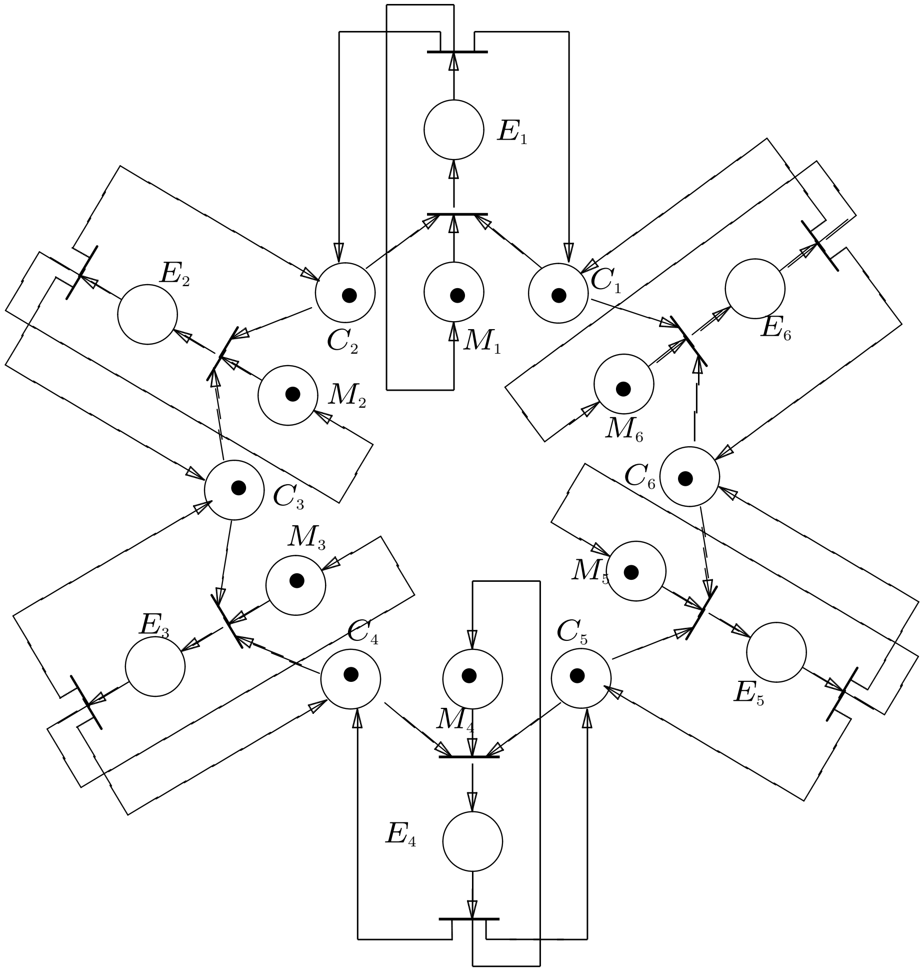
Producer/Consumer Relation



Producer/Consumer with Fixed Buffer Size



Dining Philosophers



C_i : Chop sticks
M_i : Philosopher meditating
E_i : Philosopher eating

Analysis Methods for Petri Nets

- Boundedness
- Conservation
- Liveness
- Coverability
- Persistence
- Coverability Tree

Boundedness

Definition: A place $p \in P$ in a Petri net $N = (P, T, A, w, \vec{x}_0)$ is k-bounded or k-safe if for all

$$\vec{y} \in R(\vec{x}_0) : y(p) \leq k.$$

The Petri net is called k-bounded or k-safe if all places $p \in P$ are k-bounded.

Conservation

Definition: A Petri net $N = (P, T, A, w, \vec{x}_0)$ is strictly conservative if for all $\vec{y} \in R(\vec{x}_0)$,

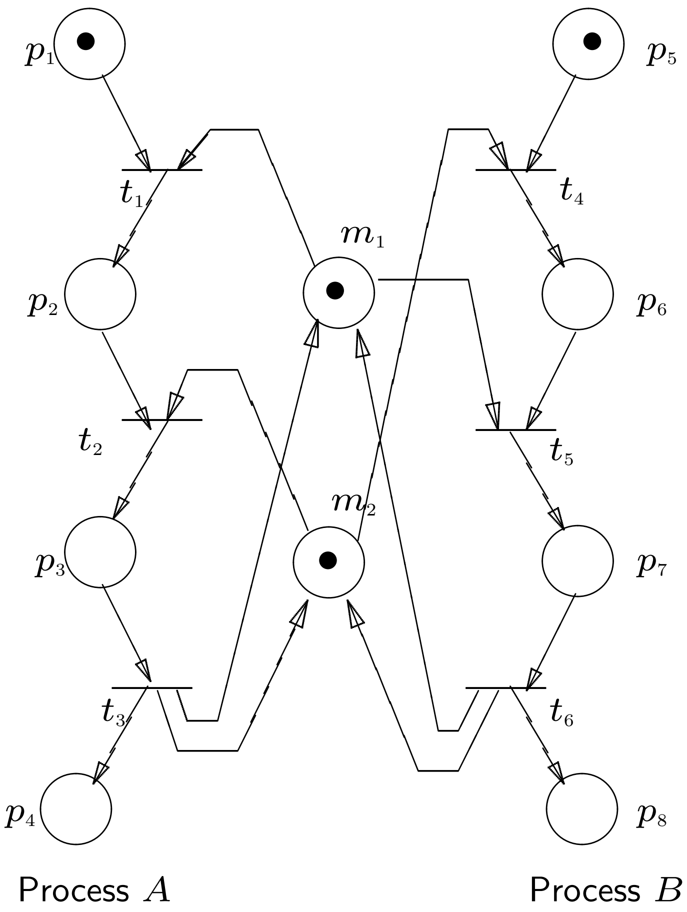
$$\sum_{p \in P} y(p) = \sum_{p \in P} x_0(p).$$

A Petri net $N = (P, T, A, w, \vec{x}_0)$ with n places is conservative with respect to a weighting vector $\vec{\gamma} = [\gamma_1, \gamma_2, ..., \gamma_n], \gamma_i \in \mathbb{N}$, if

$$\sum_{i=1}^{n} \gamma_i x(p) = \text{constant for all } p \in P \text{ and } \vec{x} \in R(\vec{x}_0).$$

The Petri net is conservative if it is conservative with respect to a weighting vector which has a positive non zero weight for all places.

Deadlock



Liveness

Definition: Let $N = (P, T, A, w, \vec{x}_0)$ be a Petri net and $\vec{x}$ a state reachable from $\vec{x}_0$.

L0-live: A transition t is live at level 0 in state $\vec{x}$ if it cannot fire in any state reachable from $\vec{x}$, i.e. it is deadlocked. .

L1-live: A transition t is live at level 1 in state $\vec{x}$ if it is potentially fire-able, i.e. if there exists a $\vec{y} \in R(\vec{x})$ such that t is enabled in $\vec{y}$.

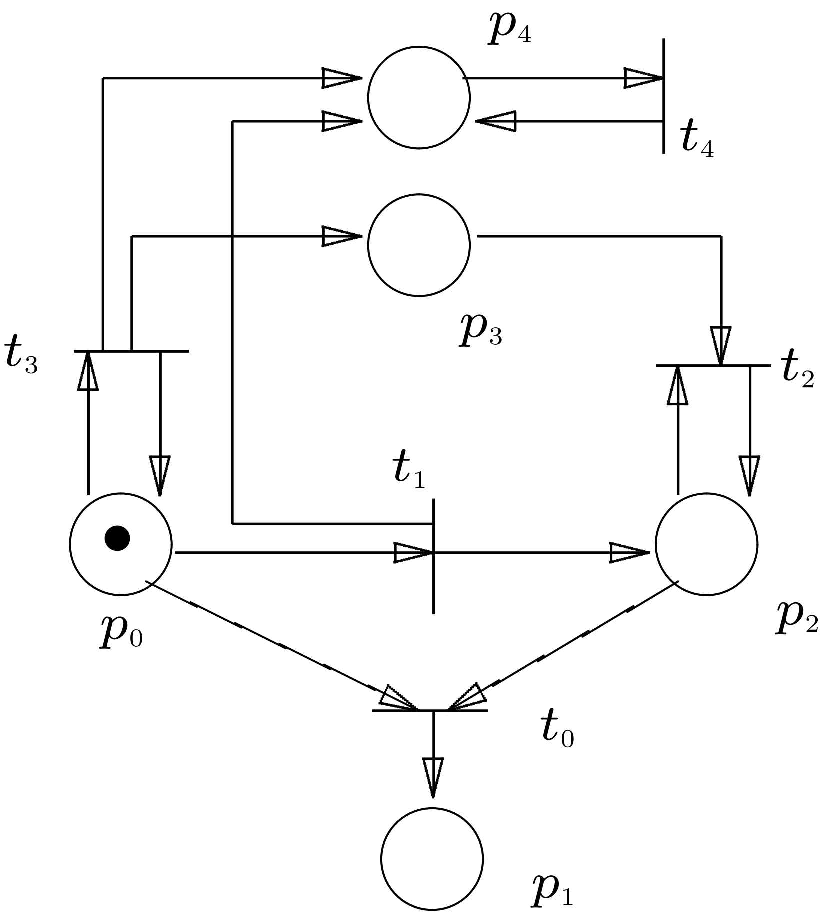
L2-live: A transition t is live at level 2 in state $\vec{x}$ if for every integer n there exists a firing sequence in which t occurs at least n times.

L3-live: A transition t is live at level 3 in state $\vec{x}$ if there is an infinite firing sequence in which t occurs infinitely often.

L4-live: A transition t is live at level 4 in state $\vec{x}$ if it is L1-live in every $\vec{y} \in R(\vec{x})$.

A Petri net is live at level i if every transition is live at level i.

Liveness Example



- t_0 is dead;
- t_1 is L1-live;
- t_2 is L2-live;
- t_3 is L3-live;
- t_4 is L4 live.

Coverability

Definition: $N = (P, T, A, w, \vec{x}_0)$ is a Petri net; $\vec{x}$ and $\vec{y}$ are arbitrary states;

State $\vec{x}$ covers state $\vec{y}$ if in $\vec{x}$ at least all transitions are enabled which are enabled in $\vec{y}$:

$$x(p) \geq y(p) \forall p \in P.$$

State $\vec{x}$ strictly covers state $\vec{y}$ if $\vec{x}$ covers $\vec{y}$ and, in addition,

$$\exists p \in P : x(p) > y(p).$$

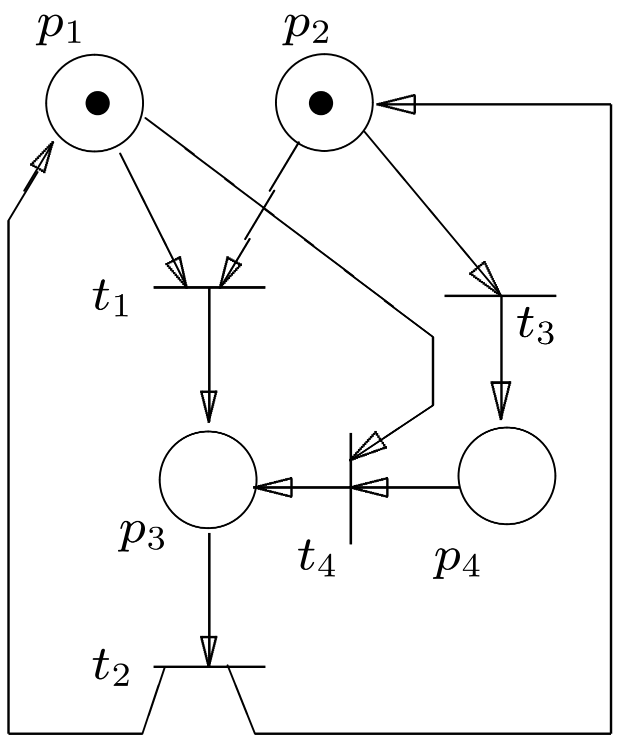
Let $\vec{x} \in R(\vec{x}_0)$. A state $\vec{y}$ is coverable by $\vec{x}$ iff there exists a state $\vec{x}' \in R(\vec{x})$ such that $x'(p) \geq y(p)$ for all $p \in P$.

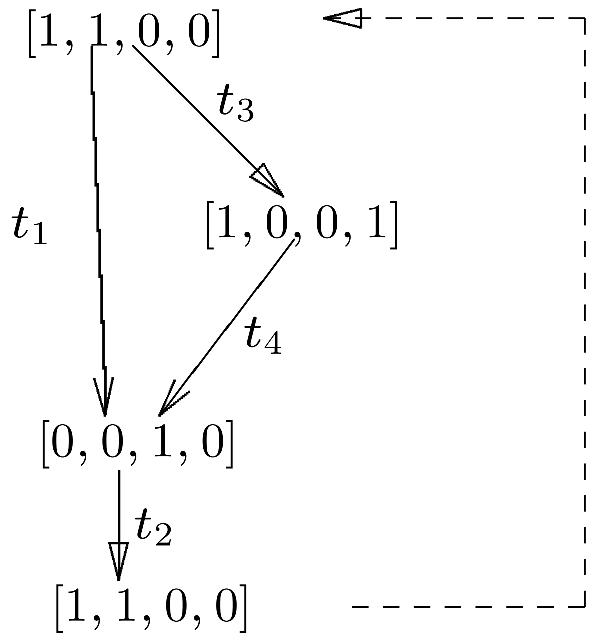
Persistence

Definition: Two transitions are persistent with respect to each other if, when both are enabled the firing of one does not disable the other.

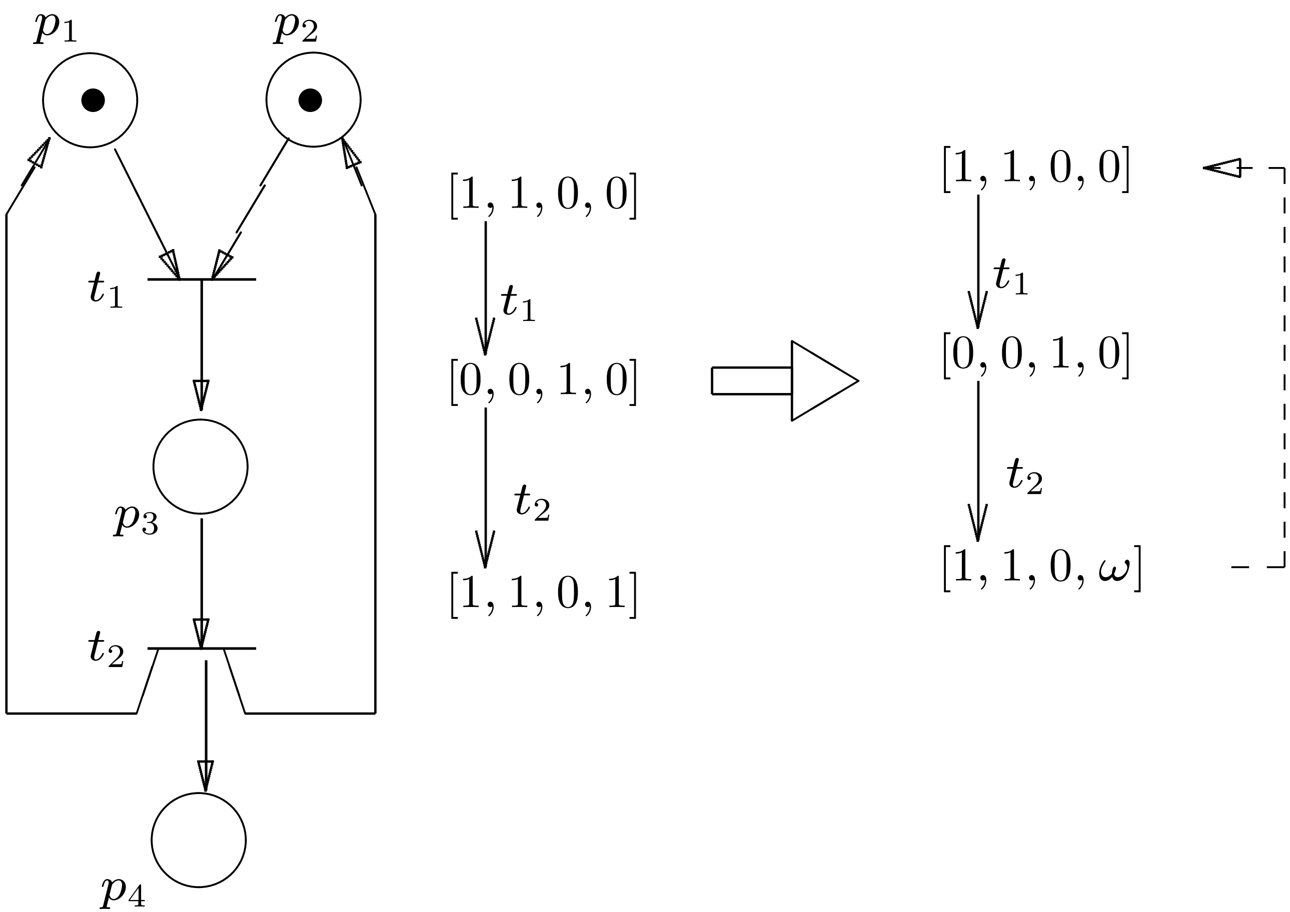
A Petri net is persistent if any two transitions are persistent with respect to each other.

Coverability Tree for a Finite State Space





Coverability Tree for an Infinite State Space



Coverability Tree Definition

Definition: Let $N = (P, T, A, w, \vec{x}_0)$ be a Petri.

A coverability tree is a tree where the arcs denote transitions $t \in T$ and the nodes represent ω-enhanced states of the Petri net.

The root node of the tree is $\vec{x}_0$.

A terminal node is an ω-enhanced state in which no transition is enabled.

A duplicate node is an ω-enhanced state which already exists somewhere else in the coverability tree.

An arc t connects two nodes $\vec{x}$ and $\vec{y}$ in the tree, iff firing of t in state $\vec{x}$ leads to state $\vec{y}$.

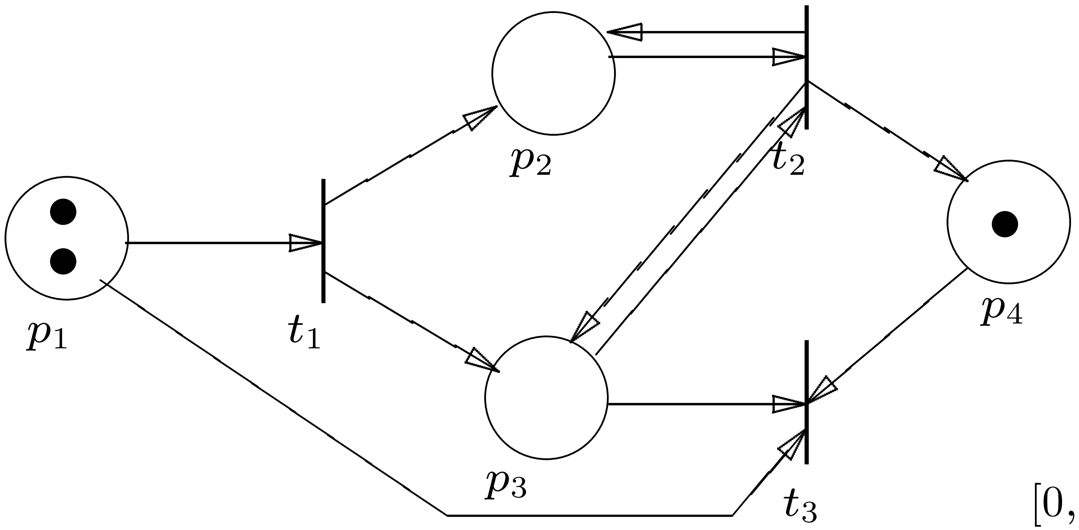
Coverability Tree Algorithm

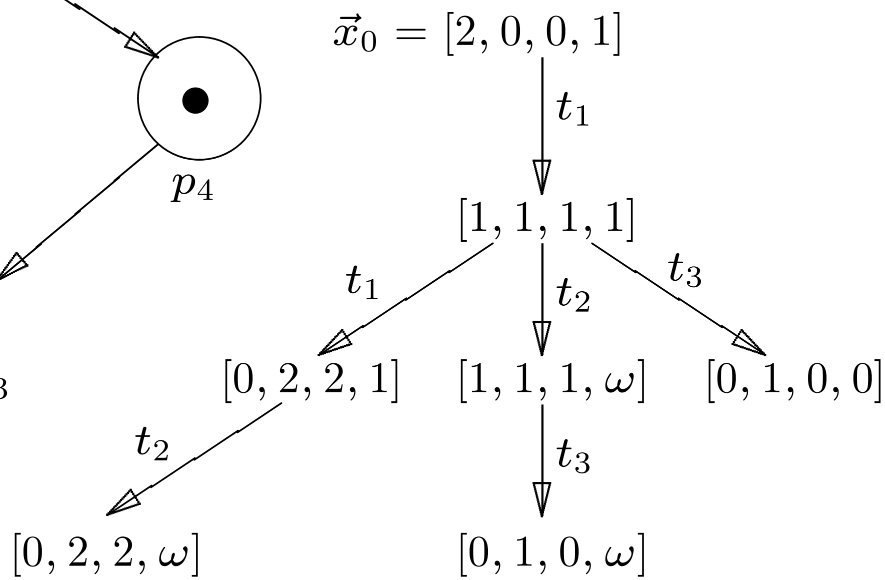
Given is the Petri net $N = (P, T, A, w, \vec{x}_0)$.

Algorithm:

Step 1. Set L, the list of open nodes, to $L := \{\vec{x}_0\}$.
Step 2. Take one node from L, named $\vec{x}$, and remove it from L;
Step 2.1. `if` $G(\vec{x}, t) = \vec{x} \ \ \forall t \in T$
`then` $\vec{x}$ is a terminal node `goto` Step 3;
Step 2.2. `for all` $\vec{x}' \in G(\vec{x}, t), t \in T, \vec{x} \neq \vec{x}'$
Step 2.2.1. `do if` $x(p) = \omega$ `then` set $x'(p) := \omega$;
Step 2.2.2. `if` there is a node $\vec{y}$ already in the tree, such that $\vec{x}$' covers $\vec{y}$
and there is a path from $\vec{y}$ to $\vec{x}$',
`then` set $x'(p) := \omega$ for all p for which $x'(p) > y(p)$;
Step 2.2.3. `if` $\vec{x}$' is not a duplicate node `then` $L := L \cup \{\vec{x}'\}$;
Step 3. `if` L is not empty `then goto` Step 2.

Coverability Tree Example



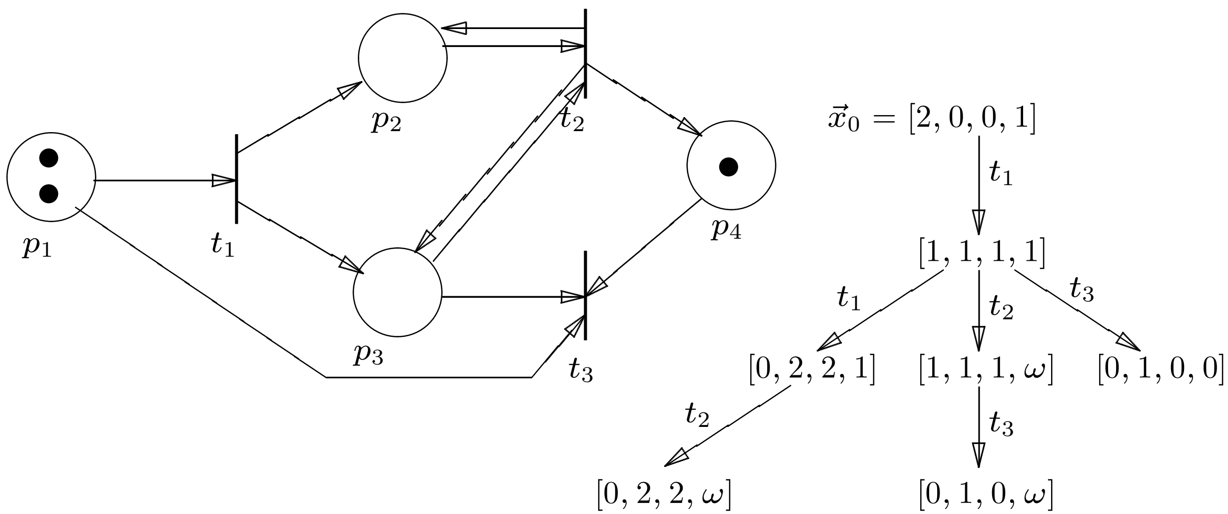


Coverability Tree: Safeness and Boundedness

- A Petri net can be k-bounded if the ω symbol never appears in its coverability tree.
- If the coverability tree contains an ω, a transition cycle to exceed a given k-bound can be identified.
- The coverability tree does not inform about the number of cycles required.

Coverability Tree: Conservation

- Recall: $\sum_{i=1}^{n} \gamma_i x(p) =$ constant for all $p \in P$ and $\vec{x} \in R(\vec{x}_0)$.
- If there is an ω the corresponding γ_i must be 0.
- We evaluate the the weighted sum for every node in the coverability tree. The net is conservative iff the result is the same for all nodes.



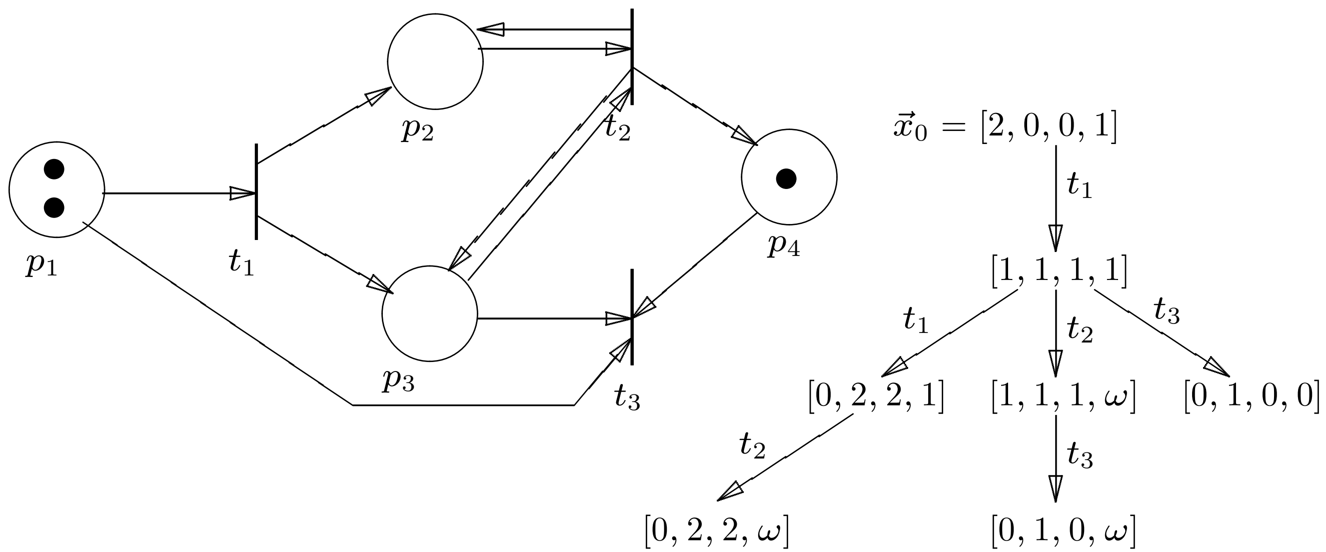
conservative for $\vec{\gamma} = [2, 3, 1, 0]$?

Computing the Conservation Vector

- Set $\gamma_i = 0$ for every unbounded place p_i.
- For b bounded places and r nodes in the coverability tree we set up r equations with $b+1$ unknown variables

$$\sum_{i=1}^{r} \gamma_i x(p_i) = C.$$

Computing the Conservation Vector - Example



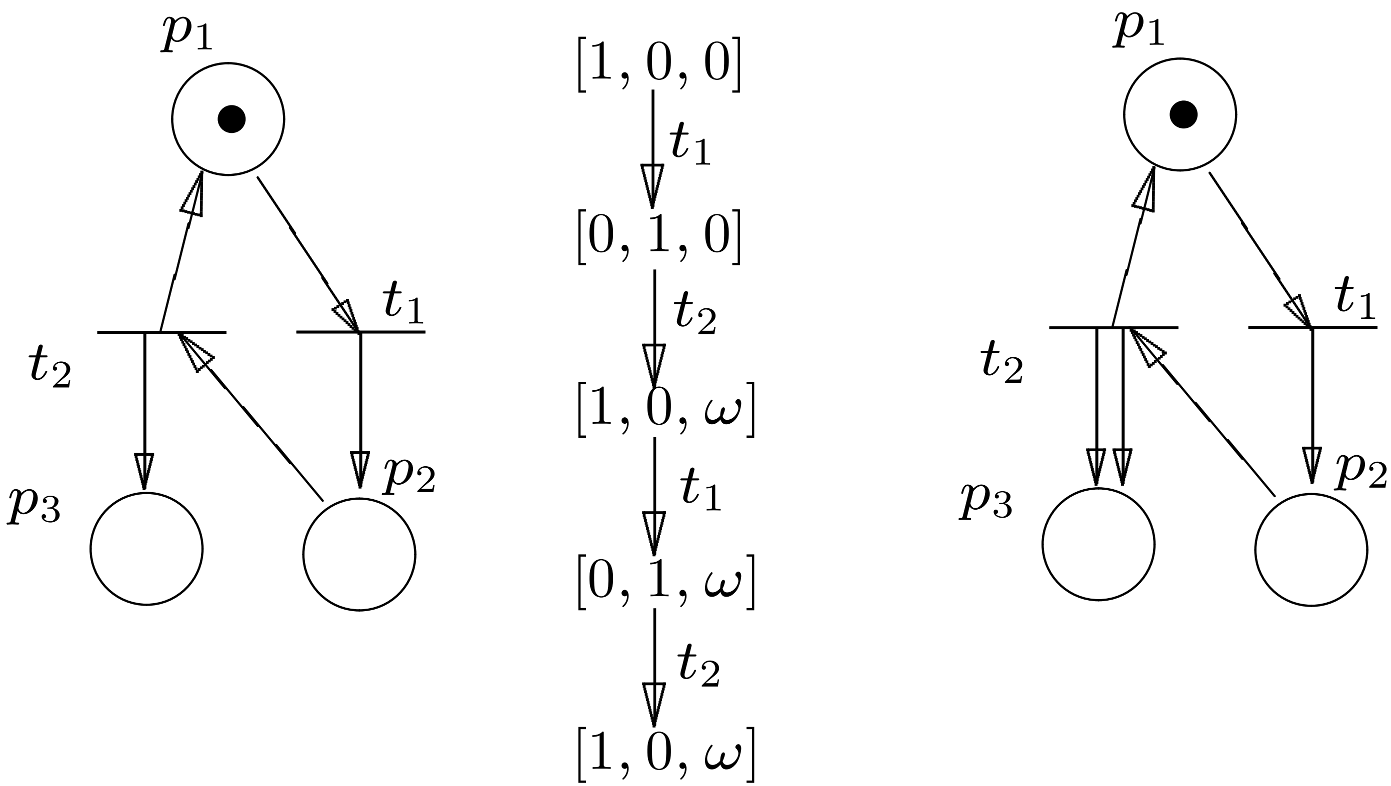
$$
\begin{array}{rcl}
\gamma_4 & = & 0 \\
2\gamma_1 + 0\gamma_2 + 0\gamma_3 & = & C \\
1\gamma_1 + 1\gamma_2 + 1\gamma_3 & = & C \\
0\gamma_1 + 2\gamma_2 + 2\gamma_3 & = & C \\
0\gamma_1 + 1\gamma_2 + 0\gamma_3 & = & C
\end{array}
$$

Only non-negative solution is $\vec{\gamma} = [0, 0, 0, 0]$ and $C = 0$.

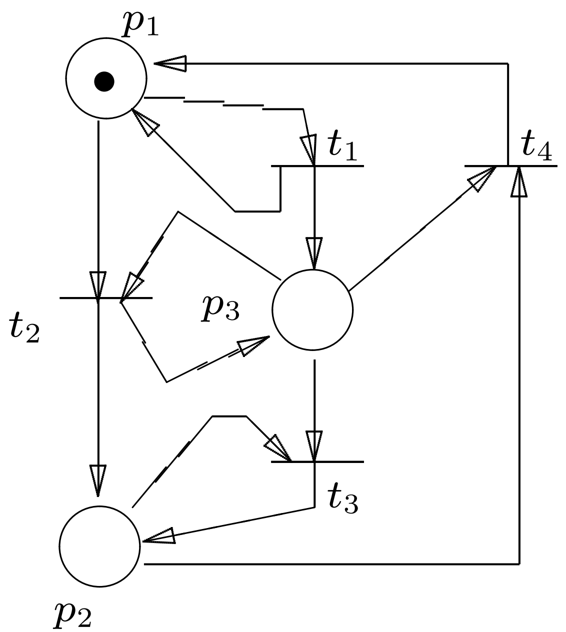
Coverability Tree: Coverability and Reachability

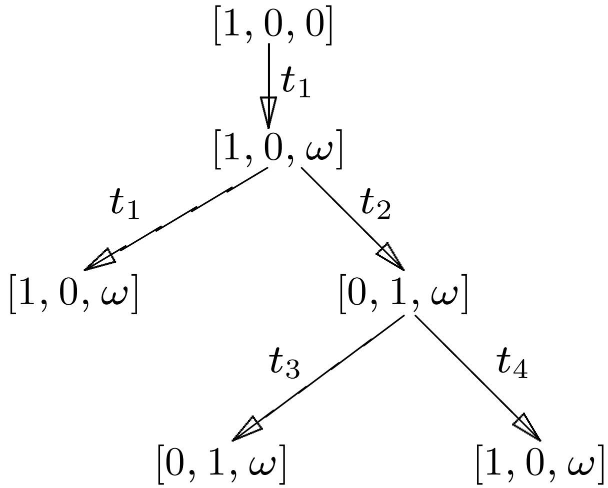
- The coverability problem can be solved by inspection of the coverability tree.
- The shortest transition sequence leading to a covering state can be found efficiently.
- The reachability problem cannot be solved in general.

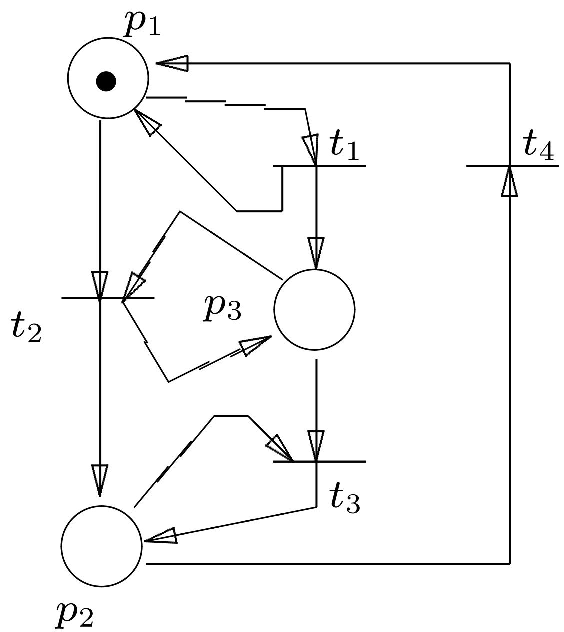
Distinct Petri Nets with Identical Coverability Tree - 1



Distinct Petri Nets with Identical Coverability Tree - 2







Can deadlock after t_1, t_2, t_3.

Cannot deadlock.

Summary

- Petri Net Dynamics
- Reachability Set
- Model Patterns
  - Server
  - Composition
  - Fork-Join
  - Conflict
  - Mutual Exclusion
  - Consumer-Producer
- Analysis Problems
  - Boundedness
  - Conservation
  - Liveness
  - Coverability
  - Persistence
  - Dead-lock
- Coverability Tree